Verification Test Problems Cardiac Electrophysiology Modeling Software
Instructions For Use

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1 Introduction

This document provides a set of test problems that can be used for verification of cardiac modeling software, i.e., to test that the model has been implemented in software correctly. These test problems are a Regulatory Science Tool created by FDA's Office of Science and Engineering Laboratories (OSEL) within the Center for Devices and Radiological Health (CDRH).

Cardiac electrophysiological or electro-mechanical simulation software typically solve well-established equations that govern the propagation of electrical waves through the heart. One activity required for demonstrating the credibility of a computational model is code verification [1]. Code verification is the process of determining if a mathematical model, and algorithms for solving the model, have been correctly implemented in the software. However, code verification is challenging with complex models. The tool provides a set of test problems with known analytic solutions, which cardiac model developers can solve using their software and thereby test if the electrophysiology modeling software has been implemented correctly.

Test problems are provided for the monodomain, bidomain and bidomain-with-bath equations. The monodomain and bidomain equations are sets of partial differential equations coupled to ordinary differential equations that have been used for many decades to model electrical activity in the heart. The bidomain-with-bath equations are a related set of equations which govern electrical fields generated in the heart and surrounding torso.

Nine test problems are provided in this document, for testing the following computational models: monodomain in 1D, 2D and 3D; bidomain in 1D, 2D and 3D; bidomain-with-bath in 1D, 2D, and 3D. Values of each of the following is specified in each test problem:

- The geometrical domain
- Tissue conductivities, surface-area-to-volume ratio, capacitance
- Sub-model of cellular dynamics
- Initial conditions
- Boundary conditions
- Stimulus current (zero)

Also provided for each test problem is:

- Exact analytic solution of test problem.

The user (cardiac model developer) should specify each of the above inputs in their software, solve the model, and compare their solution with the exact solution provided. They can then confirm the correct implementation of their software by verifying that the error converges to zero at the expected convergence rate as the spatial and temporal discretization parameters are reduced.

All test problems where originally published in [6]; see that article for background information and discussion. Note that the bidomain-with-bath problems are all essentially 1D problems, in that the solution is dependent on x only, and not y or z, regardless of dimension. Construction of genuinely 2D/3D model problems for the bidomain-with-bath model is an open problem. See [6] for results of evaluating the cardiac solver Chaste [4] using these test problems, including error norms and theoretical orders of convergence appropriate for the numerical schemes used in Chaste. Also see [3] for a second example of a cardiac solver tested using these test problems.

2 Mathematical models

Let Ω denote the geomerical domain, with boundary $\partial\Omega$. The **monodomain equations** are a set of differential equations governing the propagation of electrical waves through excitable tissue. They are a simplification of the bidomain equations (below) under the assumption that intra- and extra-cellular conductivities are proportional. The monodomain equations are [2]:

$$\chi\left(\mathcal{C}_m\frac{\partial V}{\partial t}+I_{\text{ion}}(\mathbf{u},V)\right)-\nabla\cdot(\sigma\boldsymbol{\nabla}V) = I^{(\text{stim})}, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{f}(\mathbf{u},V), \tag{2}$$

where $V\equiv V(t,\mathbf{x})$ is the transmembrane voltage, $\mathcal{C}_m$ is the specific capacitance of the cell membrane, χ is the membrane surface-area-to-volume ratio, σ is the bulk conductivity and $I^{(\text{stim})}$ is a stimulus current. $\mathbf{u}\equiv\mathbf{u}(t,\mathbf{x})$ is a vector of state variables representing the current state of the cell at location $\mathbf{x}$, and I_{ion} and $\mathbf{f}$ are prescribed functions, which together make up the cell model. Typical boundary conditions are

$$\mathbf{n}\cdot(\sigma\boldsymbol{\nabla}V)=0 \qquad \text{on } \partial\Omega, \tag{3}$$

where $\mathbf{n}$ is the outward-pointing unit normal vector. The system of equations (1)–(3) is then completed by specifying suitable initial conditions for V and $\mathbf{u}$.

The **bidomain equations** govern the propagation of the transmembrane voltage V and the extracellular potential ϕ_e. They are [2]:

$$\chi\left(\mathcal{C}_m\frac{\partial V}{\partial t}+I_{\text{ion}}(\mathbf{u},V)\right)-\nabla\cdot\left(\sigma_i\boldsymbol{\nabla}\left(V+\phi_e\right)\right) = I_i^{(\text{stim})}, \tag{4}$$

$$\nabla\cdot\left(\left(\sigma_i+\sigma_e\right)\boldsymbol{\nabla}\phi_e+\sigma_i\boldsymbol{\nabla}V\right) = -I_{\text{total}}^{(\text{stim})}, \tag{5}$$

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{f}(\mathbf{u},V), \tag{6}$$

where σ_i and σ_e are intra- and extra-cellular conductivity tensors, $I_i^{(\text{stim})}$ is an intra-cellular volume stimulus current per unit volume, and $I_{\text{total}}^{(\text{stim})}=I_i^{(\text{stim})}+I_e^{(\text{stim})}$, where $I_e^{(\text{stim})}$ is an extra-cellular volume stimulus, usually implicitly chosen so that $I_{\text{total}}^{(\text{stim})}=0$. Typical boundary conditions for (4) and (5) are the specification of zero current across the boundary:

$$\mathbf{n}\cdot\left(\sigma_i\boldsymbol{\nabla}\left(V+\phi_e\right)\right) = 0 \qquad \text{on } \partial\Omega \tag{7}$$

$$\mathbf{n}\cdot\left(\sigma_e\boldsymbol{\nabla}\phi_e\right) = 0 \qquad \text{on } \partial\Omega \tag{8}$$

As with the monodomain equations, initial conditions for V and $\mathbf{u}$ need to be specified. ϕ_e does not require initial conditions and is only determined up to a constant function of time.

Finally, the **bidomain-with-bath equations** are used to model the case of cardiac tissue contained in a conductive bath (for example, the human torso). Let Ω_b represent the bath domain, assumed to surround Ω. V is defined only in Ω (i.e. only in the tissue) and (4)–(6) still hold within Ω. However ϕ_e is now defined everywhere on $\Omega\cup\Omega_b$ (i.e. throughout the tissue and the bath), and outside the tissue satisfies

$$\nabla\cdot(\sigma_b\boldsymbol{\nabla}\phi_e)=0 \quad \text{in } \Omega_b, \tag{9}$$

where σ_b is the conductivity of the bath (usually a scalar). The boundary/interface conditions are: (7) (zero flux of ϕ_i across $\partial\Omega$); continuity of ϕ_e across $\partial\Omega$; and continuity of the extracellular current across $\partial\Omega$—the extracellular current flowing out of the tissue, $\sigma_e\nabla\phi_e\cdot\mathbf{n}$, should be equal to that entering the bath, $\sigma_b \nabla \phi_e \cdot \mathbf{n}$, everywhere on $\partial\Omega$. The remaining boundary condition on the edge of the bath domain is:

$$\mathbf{n} \cdot (\sigma_b \boldsymbol{\nabla} \phi_e) = I_E^{(\mathrm{surf})} \qquad \text{on } \partial\Omega_b \backslash \partial\Omega. \tag{10}$$

where $I_E^{(\mathrm{surf})}$ is a stimulus current (per unit area) applied to the edge of the bath domain, and may be used to represent defibrillating electrodes. The prescribed function $I_E^{(\mathrm{surf})}$ should satisfy $\int I_E^{(\mathrm{surf})}\, \mathrm{d}S = 0$ for a solution to exist (conservation of current), or alternatively a Dirichlet boundary condition on ϕ_e should be applied somewhere on $\partial\Omega_b \backslash \partial\Omega$, corresponding to a ground electrode [5].

The following non-physiological, three-variable cell model has been constructed for use in all the model problems: $\mathbf{u} = (u_1, u_2, u_3)$, specified by

$$\mathbf{f}(\mathbf{u}, V) = \begin{bmatrix} (u_1 + u_3 - V)^2 u_2^2 + \frac{1}{2}(u_1 + u_3 - V) u_2^2 (V - u_3) \\ -(u_1 + u_3 - V) u_2^3 \\ 0 \end{bmatrix} \tag{11}$$

$$I_{\mathrm{ion}}(\mathbf{u}, V) = -\frac{\mathcal{C}_m}{2}(u_1 + u_3 - V) u_2^2 (V - u_3) + \frac{\beta(V - u_3)}{\chi} \tag{12}$$

where β is a free parameter.

3 Test problems with exact solutions

3.1 Test problems for monodomain solvers

3.1.1 Monodomain in 1D

Let $F(\mathbf{x}) = \cos(\pi x)$ and $G(\mathbf{x}) = 1 + x$. Solve the monodomain equations (1)–(2) using the following inputs:

- Domain: $\Omega = [0, 1]$.
- Surface-area-to-volume ratio: $\chi = 3$
- Capacitance: $\mathcal{C}_m = 2$
- Bulk conductivity: $\sigma = 1.1 \times \pi^{-2}$
- Cell model: (11)–(12) using $\beta = -1.1$
- Stimulus current: $I^{(\text{stim})} = 0$
- Initial conditions: $V(0, \mathbf{x}) = F(\mathbf{x})$ and $\mathbf{u}(0, \mathbf{x}) = (G(\mathbf{x}) + F(\mathbf{x}), G(\mathbf{x})^{-1/2}, 0)$
- Boundary conditions: zero flux (3)

The exact solution is:

$$\begin{aligned}
V(t, \mathbf{x}) &= (1+t)^{1/2} F(\mathbf{x}) \\
u_1(t, \mathbf{x}) &= (1+t) G(\mathbf{x}) + (1+t)^{1/2} F(\mathbf{x}) \\
u_2(t, \mathbf{x}) &= (1+t)^{-1} \left(G(\mathbf{x})\right)^{-1/2} \\
u_3(t, \mathbf{x}) &= 0
\end{aligned}$$

3.1.2 Monodomain in 2D

Let $F(\mathbf{x}) = \cos(\pi x)\cos(2\pi y)$ and $G(\mathbf{x}) = 1 + xy^2$. Solve the monodomain equations (1)–(2) using the following inputs:

- Domain: $\Omega = [0,1] \times [0,1]$
- Surface-area-to-volume ratio: $\chi = 3$
- Capacitance: $\mathcal{C}_m = 2$
- Bulk conductivity: $\sigma = \pi^{-2} \begin{bmatrix} 1.1 & 0 \\ 0 & 1.2 \end{bmatrix}$
- Cell model: (11)–(12) using $\beta = -5.9$
- Stimulus current: $I^{(\text{stim})} = 0$
- Initial conditions: $V(0,\mathbf{x}) = F(\mathbf{x})$ and $\mathbf{u}(0,\mathbf{x}) = (G(\mathbf{x}) + F(\mathbf{x}), G(\mathbf{x})^{-1/2}, 0)$
- Boundary conditions: zero flux (3)

The exact solution is:

$$\begin{aligned} V(t,\mathbf{x}) &= (1+t)^{1/2} F(\mathbf{x}) \\ u_1(t,\mathbf{x}) &= (1+t)G(\mathbf{x}) + (1+t)^{1/2} F(\mathbf{x}) \\ u_2(t,\mathbf{x}) &= (1+t)^{-1} \left(G(\mathbf{x})\right)^{-1/2} \\ u_3(t,\mathbf{x}) &= 0 \end{aligned}$$

3.1.3 Monodomain in 3D

Let $F(\mathbf{x}) = \cos(\pi x)\cos(2\pi y)\cos(3\pi z)$ and $G(\mathbf{x}) = 1 + xy^2z^3$. Solve the monodomain equations (1)–(2) using the following inputs:

- Domain: $\Omega = [0,1] \times [0,1] \times [0,1]$
- Surface-area-to-volume ratio: $\chi = 3$
- Capacitance: $\mathcal{C}_m = 2$
- Bulk conductivity: $\sigma = \pi^{-2} \begin{bmatrix} 1.1 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}$
- Cell model: (11)–(12) using $\beta = -8.6$
- Stimulus current: $I^{(\text{stim})} = 0$
- Initial conditions: $V(0,\mathbf{x}) = F(\mathbf{x})$ and $\mathbf{u}(0,\mathbf{x}) = (G(\mathbf{x}) + F(\mathbf{x}), G(\mathbf{x})^{-1/2}, 0)$
- Boundary conditions: zero flux (3)

The exact solution is:

$$\begin{aligned} V(t,\mathbf{x}) &= (1+t)^{1/2}F(\mathbf{x}) \\ u_1(t,\mathbf{x}) &= (1+t)G(\mathbf{x}) + (1+t)^{1/2}F(\mathbf{x}) \\ u_2(t,\mathbf{x}) &= (1+t)^{-1}\left(G(\mathbf{x})\right)^{-1/2} \\ u_3(t,\mathbf{x}) &= 0 \end{aligned}$$

3.2 Test problems for bidomain solvers

3.2.1 Bidomain in 1D

Let $F(\mathbf{x}) = \cos(\pi x)$, $G(\mathbf{x}) = 1 + x$ and $k = 1/\sqrt{2}$. Solve the bidomain equations (4)–(6) using the following inputs:

- Domain: $\Omega = [0, 1]$.
- Surface-area-to-volume ratio: $\chi = 3$
- Capacitance: $\mathcal{C}_m = 2$
- Intracellular conductivity, $\sigma_i = 1.1 \times \pi^{-2}$
- Extracellular conductivity: $\sigma_e = (1 - k)\sigma_i/k$
- Cell model: (11)–(12) using $\beta = -1.1(1 - k)$
- Stimulus currents: $I_i^{(\mathrm{stim})} = I_{\mathrm{total}}^{(\mathrm{stim})} = 0$
- Initial conditions: $V(0, \mathbf{x}) = F(\mathbf{x})$ and $\mathbf{u}(0, \mathbf{x}) = (G(\mathbf{x}) + F(\mathbf{x}), G(\mathbf{x})^{-1/2}, 0)$
- Boundary conditions: zero flux (7)–(8)

The exact solution is:

$$
\begin{aligned}
V(t, \mathbf{x}) &= (1 + t)^{1/2} F(\mathbf{x}) \\
\phi_e(t, \mathbf{x}) &= -k(1 + t)^{1/2} F(\mathbf{x}) + C(t) \\
u_1(t, \mathbf{x}) &= (1 + t)G(\mathbf{x}) + (1 + t)^{1/2} F(\mathbf{x}) \\
u_2(t, \mathbf{x}) &= (1 + t)^{-1} \left(G(\mathbf{x})\right)^{-1/2} \\
u_3(t, \mathbf{x}) &= 0
\end{aligned}
$$

where $C(t)$ is an arbitrary function of time. The specific implementation of the bidomain equations in the software tested will determine $C(t)$ in the simulated ϕ_e. For example, a numerical scheme that imposes zero mean, $\int_\Omega \phi_e \, \mathrm{d}^3\mathbf{x} = 0$, implies that $C(t) = 0$.

3.2.2 Bidomain in 2D

Let $F(\mathbf{x}) = \cos(\pi x)\cos(2\pi y)$, $G(\mathbf{x}) = 1 + xy^2$ and $k = 1/\sqrt{2}$. Solve the bidomain equations (4)–(6) using the following inputs:

- Domain: $\Omega = [0,1] \times [0,1]$
- Surface-area-to-volume ratio: $\chi = 3$
- Capacitance: $\mathcal{C}_m = 2$
- Intracellular conductivity: $\sigma_i = \pi^{-2} \begin{bmatrix} 1.1 & 0 \\ 0 & 1.2 \end{bmatrix}$
- Extracellular conductivity: $\sigma_e = (1-k)\sigma_i/k$
- Cell model: (11)–(12) using $\beta = -5.9(1-k)$
- Stimulus currents: $I_i^{(\text{stim})} = I_{\text{total}}^{(\text{stim})} = 0$
- Initial conditions: $V(0,\mathbf{x}) = F(\mathbf{x})$ and $\mathbf{u}(0,\mathbf{x}) = (G(\mathbf{x}) + F(\mathbf{x}), G(\mathbf{x})^{-1/2}, 0)$
- Boundary conditions: zero flux (7)–(8)

The exact solution is:

$$\begin{aligned}
V(t,\mathbf{x}) &= (1+t)^{1/2}F(\mathbf{x}) \\
\phi_e(t,\mathbf{x}) &= -k(1+t)^{1/2}F(\mathbf{x}) + C(t) \\
u_1(t,\mathbf{x}) &= (1+t)G(\mathbf{x}) + (1+t)^{1/2}F(\mathbf{x}) \\
u_2(t,\mathbf{x}) &= (1+t)^{-1}\left(G(\mathbf{x})\right)^{-1/2} \\
u_3(t,\mathbf{x}) &= 0
\end{aligned}$$

where $C(t)$ is an arbitrary function of time. The specific implementation of the bidomain equations in the software tested will determine $C(t)$ in the simulated ϕ_e. For example, a numerical scheme that imposes zero mean, $\int_\Omega \phi_e \, \mathrm{d}^3\mathbf{x} = 0$, implies that $C(t) = 0$.

3.2.3 Bidomain in 3D

Let $F(\mathbf{x}) = \cos(\pi x)\cos(2\pi y)\cos(3\pi z)$, $G(\mathbf{x}) = 1 + xy^2z^3$ and $k = 1/\sqrt{2}$. Solve the bidomain equations (4)–(6) using the following inputs:

- Domain: $\Omega = [0,1] \times [0,1] \times [0,1]$
- Surface-area-to-volume ratio: $\chi = 3$
- Capacitance: $\mathcal{C}_m = 2$
- Cell model: (11)–(12) using $\beta = -8.6(1-k)$
- Intracellular conductivity: $\sigma_i = \pi^{-2} \begin{bmatrix} 1.1 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}$
- Extracellular conductivity: $\sigma_e = (1-k)\sigma_i/k$
- Stimulus currents: $I_i^{(\text{stim})} = I_{\text{total}}^{(\text{stim})} = 0$
- Initial conditions: $V(0,\mathbf{x}) = F(\mathbf{x})$ and $\mathbf{u}(0,\mathbf{x}) = (G(\mathbf{x}) + F(\mathbf{x}), G(\mathbf{x})^{-1/2}, 0)$
- Boundary conditions: zero flux (7)–(8)

The exact solution is:

$$
\begin{aligned}
V(t,\mathbf{x}) &= (1+t)^{1/2} F(\mathbf{x}) \\
\phi_e(t,\mathbf{x}) &= -k(1+t)^{1/2} F(\mathbf{x}) + C(t) \\
u_1(t,\mathbf{x}) &= (1+t)G(\mathbf{x}) + (1+t)^{1/2} F(\mathbf{x}) \\
u_2(t,\mathbf{x}) &= (1+t)^{-1} \left(G(\mathbf{x})\right)^{-1/2} \\
u_3(t,\mathbf{x}) &= 0
\end{aligned}
$$

where $C(t)$ is an arbitrary function of time. The specific implementation of the bidomain equations in the software tested will determine $C(t)$ in the simulated ϕ_e. For example, a numerical scheme that imposes zero mean, $\int_\Omega \phi_e \, \mathrm{d}^3\mathbf{x} = 0$, implies that $C(t) = 0$.

3.3 Test problems for bidomain-with-bath solvers

3.3.1 Bidomain-with-bath in 1D

Let $F(\mathbf{x}) = \cos(\pi x)$, $G(\mathbf{x}) = 1 + x$, $k = 1/\sqrt{2}$ and $\alpha = 0.01$. Solve the bidomain-with-bath model using the following inputs (in the below s_e denotes σ_e):

- Using $\Omega_{all} = [-1, 2]$, $\Omega_b = \{\mathbf{x} \in \Omega_{all} : -1 \le x \le 0 \text{ or } 1 \le x \le 2\}$ and $\Omega = \{\mathbf{x} \in \Omega_{all} : 0 \le x \le 1\}$
- Surface-area-to-volume ratio: $\chi = 3$
- Capacitance: $\mathcal{C}_m = 2$
- Intracellular conductivity, $\sigma_i = 1.1 \times \pi^{-2}$
- Extracellular conductivity: $\sigma_e = (1 - k)\sigma_i/k$
- Bath conductivity: $\sigma_b = s_e/2$
- Cell model: (11)–(12) using $\beta = -1.1(1 - k)$
- Stimulus currents: $I_i^{(\text{stim})} = I_{\text{total}}^{(\text{stim})} = 0$
- Initial conditions: $V(0, \mathbf{x}) = F(\mathbf{x}) - \frac{\alpha x}{s_e}$ and $\mathbf{u}(0, \mathbf{x}) = (G(\mathbf{x}) + F(\mathbf{x}), G(\mathbf{x})^{-1/2}, -\frac{\alpha x}{s_e})$.
- External stimuli:

$$I_E^{(\text{surf})} = \begin{cases} -\alpha & \text{if } x = -1 \\ \alpha & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

The exact solution of this problem is:

$$\begin{aligned}
V(t, \mathbf{x}) &= (1+t)^{1/2} F(\mathbf{x}) - \frac{\alpha}{s_e} x \\
\phi_e(t, \mathbf{x}) &= \begin{cases} -k(1+t)^{1/2} + \frac{\alpha}{\sigma_b} x + C(t) & \text{if } -1 \le x \le 0 \\ -k(1+t)^{1/2} \cos(\pi x) + \frac{\alpha}{s_e} x + C(t) & \text{if } 0 \le x \le 1 \\ -k(1+t)^{1/2} \cos(\pi) + \frac{\alpha}{s_e} + \frac{\alpha}{\sigma_b}(x - 1) + C(t) & \text{if } 1 \le x \le 2 \end{cases} \\
u_1(t, \mathbf{x}) &= (1+t) G(\mathbf{x}) + (1+t)^{1/2} F(\mathbf{x}) \\
u_2(t, \mathbf{x}) &= (1+t)^{-1} \left(G(\mathbf{x})\right)^{-1/2} \\
u_3(t, \mathbf{x}) &= -\frac{\alpha}{s_e} x
\end{aligned}$$

where $C(t)$ is an arbitrary function of time.

Ground electrode variant: as above except with the Dirichlet boundary $\phi_e = 0$ on $x = -1$, with $I_E^{(\text{surf})} = \alpha$ on $x = 2$ and $I_E^{(\text{surf})} = 0$ on the other boundaries. Then the exact solution is as above with $C(t) = k(1+t)^{1/2} + \frac{\alpha}{\sigma_b}$.

3.3.2 Bidomain-with-bath in 2D

Let $F(\mathbf{x}) = \cos(\pi x)$ (note: unlike the other 2D test problems, this should just be a function of x, not x and y), $G(\mathbf{x}) = 1 + xy^2$, $k = 1/\sqrt{2}$ and $\alpha = 0.01$. Solve the bidomain-with-bath model using the following inputs (in the below s_e denotes $(\sigma_e)_{11}$):

- Using $\Omega_{all} = [-1, 2] \times [0, 1]$, $\Omega_b = \{\mathbf{x} \in \Omega_{all} : -1 \le x \le 0 \text{ or } 1 \le x \le 2\}$ and $\Omega = \{\mathbf{x} \in \Omega_{all} : 0 \le x \le 1\}$
- Surface-area-to-volume ratio: $\chi = 3$
- Capacitance: $\mathcal{C}_m = 2$
- Intracellular conductivity: $\sigma_i = \pi^{-2} \begin{bmatrix} 1.1 & 0 \\ 0 & 1.2 \end{bmatrix}$
- Extracellular conductivity: $\sigma_e = (1 - k)\sigma_i / k$
- Bath conductivity: $\sigma_b = s_e/2$
- Cell model: (11)–(12) using $\beta = -1.1(1 - k)$
- Stimulus currents: $I_i^{(\text{stim})} = I_{\text{total}}^{(\text{stim})} = 0$
- Initial conditions: $V(0, \mathbf{x}) = F(\mathbf{x}) - \frac{\alpha x}{s_e}$ and $\mathbf{u}(0, \mathbf{x}) = (G(\mathbf{x}) + F(\mathbf{x}), G(\mathbf{x})^{-1/2}, -\frac{\alpha x}{s_e})$.
- External stimuli:

$$I_E^{(\text{surf})} = \begin{cases} -\alpha & \text{if } x = -1 \\ \alpha & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

The exact solution of this problem is:

$$\begin{aligned}
V(t, \mathbf{x}) &= (1 + t)^{1/2} F(\mathbf{x}) - \frac{\alpha}{s_e} x \\
\phi_e(t, \mathbf{x}) &= \begin{cases} -k(1 + t)^{1/2} + \frac{\alpha}{\sigma_b} x + C(t) & \text{if } -1 \le x \le 0 \\ -k(1 + t)^{1/2} \cos(\pi x) + \frac{\alpha}{s_e} x + C(t) & \text{if } 0 \le x \le 1 \\ -k(1 + t)^{1/2} \cos(\pi) + \frac{\alpha}{s_e} + \frac{\alpha}{\sigma_b}(x - 1) + C(t) & \text{if } 1 \le x \le 2 \end{cases} \\
u_1(t, \mathbf{x}) &= (1 + t) G(\mathbf{x}) + (1 + t)^{1/2} F(\mathbf{x}) \\
u_2(t, \mathbf{x}) &= (1 + t)^{-1} (G(\mathbf{x}))^{-1/2} \\
u_3(t, \mathbf{x}) &= -\frac{\alpha}{s_e} x
\end{aligned}$$

where $C(t)$ is an arbitrary function of time.

Ground electrode variant: as above except with the Dirichlet boundary $\phi_e = 0$ on $x = -1$, with $I_E^{(\text{surf})} = \alpha$ on $x = 2$ and $I_E^{(\text{surf})} = 0$ on the other boundaries. Then the exact solution is as above with $C(t) = k(1 + t)^{1/2} + \frac{\alpha}{\sigma_b}$.

3.3.3 Bidomain-with-bath in 3D

Let $F(\mathbf{x}) = \cos(\pi x)$ (note: unlike the other 3D test problems, this should just be a function of x, not x, y and z), let $G(\mathbf{x}) = 1 + xy^2z^3$, $k = 1/\sqrt{2}$ and $\alpha = 0.01$. Solve the bidomain-with-bath model using the following inputs (in the below s_e denotes $(\sigma_e)_{11}$):

- Using $\Omega_{all} = [-1, 2] \times [0, 1] \times [0, 1]$, $\Omega_b = \{\mathbf{x} \in \Omega_{all} : -1 \le x \le 0 \text{ or } 1 \le x \le 2\}$ and $\Omega = \{\mathbf{x} \in \Omega_{all} : 0 \le x \le 1\}$
- Surface-area-to-volume ratio: $\chi = 3$
- Capacitance: $\mathcal{C}_m = 2$
- Intracellular conductivity: $\sigma_i = \pi^{-2} \begin{bmatrix} 1.1 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}$
- Extracellular conductivity: $\sigma_e = (1 - k)\sigma_i/k$
- Bath conductivity: $\sigma_b = s_e/2$
- Cell model: (11)–(12) using $\beta = -1.1(1 - k)$
- Stimulus currents: $I_i^{(\text{stim})} = I_{\text{total}}^{(\text{stim})} = 0$
- Initial conditions: $V(0, \mathbf{x}) = F(\mathbf{x}) - \frac{\alpha x}{s_e}$ and $\mathbf{u}(0, \mathbf{x}) = (G(\mathbf{x}) + F(\mathbf{x}), G(\mathbf{x})^{-1/2}, -\frac{\alpha x}{s_e})$.
- External stimuli:

$$I_E^{(\text{surf})} = \begin{cases} -\alpha & \text{if } x = -1 \\ \alpha & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

The exact solution of this problem is:

$$\begin{aligned}
V(t, \mathbf{x}) &= (1 + t)^{1/2} F(\mathbf{x}) - \frac{\alpha}{s_e} x \\
\phi_e(t, \mathbf{x}) &= \begin{cases} -k(1 + t)^{1/2} + \frac{\alpha}{\sigma_b} x + C(t) & \text{if } -1 \le x \le 0 \\ -k(1 + t)^{1/2} \cos(\pi x) + \frac{\alpha}{s_e} x + C(t) & \text{if } 0 \le x \le 1 \\ -k(1 + t)^{1/2} \cos(\pi) + \frac{\alpha}{s_e} + \frac{\alpha}{\sigma_b}(x - 1) + C(t) & \text{if } 1 \le x \le 2 \end{cases} \\
u_1(t, \mathbf{x}) &= (1 + t)G(\mathbf{x}) + (1 + t)^{1/2} F(\mathbf{x}) \\
u_2(t, \mathbf{x}) &= (1 + t)^{-1} (G(\mathbf{x}))^{-1/2} \\
u_3(t, \mathbf{x}) &= -\frac{\alpha}{s_e} x
\end{aligned}$$

where $C(t)$ is an arbitrary function of time.

Ground electrode variant: as above except with the Dirichlet boundary $\phi_e = 0$ on $x = -1$, with $I_E^{(\text{surf})} = \alpha$ on $x = 2$ and $I_E^{(\text{surf})} = 0$ on the other boundaries. Then the exact solution is as above with $C(t) = k(1 + t)^{1/2} + \frac{\alpha}{\sigma_b}$.

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